Lax representation and Poisson geometry of the Kowalevski top

2001 J. Phys. A: Math. Gen. 34 2077

(http://iopscience.iop.org/0305-4470/34/11/301)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 2077–2085 www.iop.org/Journals/ja PII: S0305-4470(01)15496-X

Lax representation and Poisson geometry of the Kowalevski top

Gregorio Falqui

SISSA, Via Beirut 2/4, I-34014 Trieste, Italy

E-mail: falqui@sissa.it

Received 5 July 2000

Abstract

We discuss the Poisson structure underlying the two-field Kowalevski gyrostat and the Kowalevski top. We start from their Lax structure and construct a suitable pencil of Poisson brackets which endows these systems with the structure of bi-Hamiltonian completely integrable systems. We study the Casimir functions of such pencils, and show how it is possible to frame the Kowalevski systems within the so-called Gel'fand–Zakharevich bi-Hamiltonian setting for integrable systems.

PACS numbers: 0230I, 0210, 0220, 0240, 4520J

1. Introduction

The Kowalevski top (and its generalizations) are among the most intriguing finite-dimensional completely integrable Hamiltonian systems (see, e.g., [2, pp 189 ff] and references quoted therein). Framing them within the theory of Lax pairs and understanding their structure of algebraically completely integrable systems [1, 3, 6, 9] was one of the major breakthroughs in the study of these topics.

The aim of this paper is to deepen the study of some of their Hamiltonian aspects. Our starting point is the final section of [8], where it has been shown that the Lax formulation for the so-called Kowalevski gyrostat in two fields found in [9] admits both an r-matrix interpretation and a bi-Hamiltonian formulation. To achieve this, a peculiar splitting of the zero-degree part of the relevant twisted loop algebra, and an extension of the nine-dimensional Bobenko, Reyman, Semenov-Tian-Shansky phase space to a ten-dimensional bi-Hamiltonian manifold were performed.

Our first task is to show that the geometric counterpart of such a construction is a deformation of the 'standard' Poisson pencil along a triple of characteristic vector fields. Since such a deformation preserves the property that bi-Hamiltonian vector fields induce Lax flows on the Lax matrix, the commuting Hamiltonians of the deformed Poisson pencil coincide with those of the undeformed pencil (although the bi-Hamiltonian vector fields *are different*). Then we will discuss the so-called Gel'fand–Zakharevich geometry of the two-field

Kowalevski gyrostat. We shall show that it can be represented by a pair of Lenard sequences of length 2, that is, in the terminology of [5], the phase space of the two-field Kowalevski gyrostat has a pure Kronecker structure of type (5, 5).

As is well established, the original Kowalevski top sits inside the two-field Kowalevski gyrostat system as a submanifold obtained simply by putting the extra degrees of freedom to zero. Thus it is very natural to try to endow the Kowalevski top hierarchy with a Poisson pencil obtained by Dirac reduction from the original one. Indeed, at the end of the paper, we show that this procedure is effective and endows the Kowalevski top system with a pure Kronecker bi-Hamiltonian structure of type (3, 3).

2. Pencils of Poisson structures and Lax representations

In this section we will briefly discuss/review some facts concerning the relations between the Lax representation of a dynamical system defined on a space of matrices polynomially depending on the spectral parameter, and the bi-Hamiltonian representation of the system itself. We will restrict ourselves to the case of the twisted loop algebra associated with the Kowalevski problem.

Let us consider $\mathfrak{sp}(4)$ and the involution τ on the loop space $L\mathfrak{sp}(4)((z))$ defined by

$$\tau(X(z)) = -X^t(-z). \tag{2.1}$$

If the fundamental symplectic matrix J is chosen to be

$$\mathsf{J} = \begin{bmatrix} \mathrm{i}\sigma_2 & \mathbf{0} \\ \mathbf{0} & \mathrm{i}\sigma_2 \end{bmatrix} \qquad \text{with} \quad \mathrm{i}\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.2}$$

one sees that the Cartan decomposition $\mathfrak{sp}(4) = \mathfrak{a} \oplus \mathfrak{s}$ into spaces of antisymmetric and symmetric traceless matrices holds. Such a decomposition is particularly convenient to explicitly describe the twisted loop algebra $L_\tau\mathfrak{sp}(4)((z))$ formed by those elements fixed by the involution τ, since its polynomial elements are given by sums of the form

$$X(z) = \mathsf{A}_1 + z\mathsf{S}_1 + z^2\mathsf{A}_2 + z^3\mathsf{S}_2 + \cdots \qquad \mathsf{A}_i \in \mathfrak{a} \quad \mathsf{S}_i \in \mathfrak{s}. \tag{2.3}$$

As is well known (see, e.g., [7, 10]), on the loop space of an arbitrary Lie algebra $\mathfrak{g}$ a family of mutually compatible Poisson structures Π_l, associated with the family r_l of classical r-matrices

$$r_l(X(z)) = (z^l X(z))_+ - (z^l X(z))_- \tag{2.4}$$

is defined[1]. By means of straightforward calculations one proves the following.

Proposition 2.1. *Let $\mathcal{M}^3_\Sigma$ be the affine subspace of $L_\tau\mathfrak{sp}(4)((z))$ consisting of polynomial matrices of the form*

$$M = z^3\Sigma + z^2 A + zS + B \qquad \Sigma = \begin{bmatrix} 0 & 0 \\ 0 & -2\sigma_3 \end{bmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.5}$$

where $A^t = -A$, $B^t = -B$, $S^t = S$. The Poisson structures Π_1 and Π_3 restrict to $\mathcal{M}^3_\Sigma$ and form a Poisson pencil $\mathcal{P}_z = \Pi_3 - z^2\Pi_1 := \mathcal{P} - z^2\mathcal{R}$.

Explicitly, the two Poisson structures can be described as follows. Let $\frac{\partial H}{\partial A}, \frac{\partial H}{\partial S}, \frac{\partial H}{\partial B}$ be matrices representing (under the natural identification given by the trace form) the derivatives with respect to the natural coordinates in $\mathcal{M}^3_\Sigma$ of any Hamiltonian function H. The Hamiltonian vector fields with respect to $\mathcal{P}$ are given by

$$\begin{bmatrix} \dot{A} \\ \dot{S} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} [A,\cdot] & [S,\cdot] & [B,\cdot] \\ [S,\cdot] & [B,\cdot] & \\ [B,\cdot] & & \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial A} \\ \frac{\partial H}{\partial S} \\ \frac{\partial H}{\partial B} \end{bmatrix} \tag{2.6}$$

[1] Here, $(\cdot)_\pm$ refers to the projection onto non-negative and negative powers of z.

while the Hamiltonian vector fields with respect to $\mathcal{R}$ are given by

$$\begin{bmatrix} \dot{A} \\ \dot{S} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} & -[\Sigma, \cdot] & \\ -[\Sigma, \cdot] & -[A, \cdot] & \\ & & [B, \cdot] \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial A} \\ \frac{\partial H}{\partial S} \\ \frac{\partial H}{\partial B} \end{bmatrix}. \tag{2.7}$$

The fundamental property of such a formulation relies on the following property, that can be proven by a straightforward computation.

Proposition 2.2. *All vector fields that can be represented in the form*

$$\begin{bmatrix} \dot{A} \\ \dot{S} \\ \dot{B} \end{bmatrix} = \left(\mathcal{P} - z^2 \mathcal{R}\right) \begin{bmatrix} V_A \\ V_S \\ V_B \end{bmatrix} \tag{2.8}$$

where $\{V_A, V_S, V_B\}$ *stands for any (possibly z-dependent) differential one-form on* $\mathcal{M}^3_\Sigma$, *that is, in the Gel'fand–Zakharevich parlance, all vector fields that are tangent to the soul (or axis) of the bi-Hamiltonian manifold* $\mathcal{M}^3_\Sigma$, *admit the Lax representation*

$$\dot{L}(z) = [L(z), M(z)] \tag{2.9}$$

where

$$L(z) = z^3 \Sigma + z^2 A + zS + B \qquad \textit{and} \qquad M(z) = zV_S + V_A. \tag{2.10}$$

2.1. Deformation of the standard Poisson pencil and the bi-Hamiltonian structure for the two-field Kowalevski gyrostat

In this section we shall show that the Poisson pencil for the Kowalevski gyrostat found in [8], within the r-matrix theoretical framework, can be geometrically seen as the result of a peculiar deformation of the 'natural'[2] Poisson pencil given by equations (2.6) and (2.7). The fundamental observation in this respect is the following: consider the three vector fields

- X defined by $\dot{A} = \mathsf{J}$, $\dot{S} = 0$, $\dot{B} = 0$,
- $Y = P\, \mathrm{d}[A]^1_2$ ($[A]^1_2$ is the $(1, 2)$ element of the matrix A of (2.5)) and
- Z defined by $\dot{A} = 0$, $\dot{S} = 0$, $\dot{B} = \mathsf{J}$,

where J is the canonical symplectic matrix (2.2). Thanks to the fact that an antisymmetric symplectic matrix *commutes* with J, one easily sees that X is a symmetry of $\mathcal{P}$ and Z is a symmetry of $\mathcal{R}$, that is

$$\mathrm{Lie}_X(\mathcal{P}) = 0 \qquad \mathrm{Lie}_Z(\mathcal{R}) = 0. \tag{2.11}$$

Moreover, since $[A]^1_2$ is a Casimir of $\mathcal{R}$, Y is a symmetry of both Poisson tensors. This follows immediately once one rewrites the compatibility condition between $\mathcal{P}$ and $\mathcal{R}$ in the equivalent form:

$$\mathrm{Lie}_{\mathcal{P}df}\mathcal{R} + \mathrm{Lie}_{\mathcal{R}df}\mathcal{P} = 0 \qquad \forall \text{ function } f.$$

These three vector fields mutually commute and the following proposition holds.

Proposition 2.3.

$$\mathrm{Lie}_X(\mathcal{R}) + \mathrm{Lie}_Z(\mathcal{P}) = 0 \tag{2.12}$$

so that the pencil of vector fields $\mathcal{X}(z) := X + z^2 Z$ *is a* symmetry *of the Poisson pencil* $\mathcal{P}_z = \mathcal{P} - z^2\mathcal{R}$.

[2] A study of the Kowalevski gyrostat as a system endowed with undeformed brackets was performed in the preprint [11].

Proof. It follows from the observation that the Lie derivative of $\mathcal{R}$ w.r.t. X is the bivector given by

$$\begin{bmatrix} \dot{A} \\ \dot{S} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} & & 0 \\ & -[\mathsf{J},\cdot] & \\ 0 & & \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial A} \\ \frac{\partial H}{\partial S} \\ \frac{\partial H}{\partial B} \end{bmatrix} \tag{2.13}$$

and the Lie derivative of $\mathcal{P}$ w.r.t. Z is expressed by

$$\begin{bmatrix} \dot{A} \\ \dot{S} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} & & [\mathsf{J},\cdot] \\ & [\mathsf{J},\cdot] & \\ [\mathsf{J},\cdot] & & \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial A} \\ \frac{\partial H}{\partial S} \\ \frac{\partial H}{\partial B} \end{bmatrix}. \tag{2.14}$$

The validity of relation (2.12) (which immediately implies the fact that $\mathcal{X}(z)$ is a symmetry of $\mathcal{P}_z$) follows by noticing that the decomposition of $\mathfrak{sp}(4)$ into symmetric and antisymmetric matrices is an orthogonal decomposition with respect to the trace form. So, for every function H, $\frac{\partial H}{\partial A}$ and $\frac{\partial H}{\partial B}$ are *antisymmetric* matrices, and hence commute with J. □

We now consider the deformation of the Poisson pencil defined as follows: let $P = \mathcal{P} - X \wedge Y$ and $R = \mathcal{R} - Z \wedge Y$.

Proposition 2.4. *The pencil of bivectors*

$$P_z = P - z^2 R$$

endows $\mathcal{M}^3_\Sigma$ with another structure of the bi-Hamiltonian manifold. Moreover, such deformation preserves the Lax property, that is every vector tangent to the axis of P_z admits a Lax representation

$$\dot{L}(z) = [L(z), \hat{M}(z)]. \tag{2.15}$$

Proof. The first part of the assertion follows from the above-stated properties of the vector fields X, Y and Z, since they imply that the Schouten bracket $[P_z, P_z]$ vanishes. As for the second part, one can argue as follows. We have seen in proposition 2.2 that locally Hamiltonian vector fields with respect to the 'undeformed' pencil $\mathcal{P}_z$ admit a Lax representation. We have only to show that this property is preserved by the deformation of the Poisson pencil $P_z - \mathcal{P}_z = (Z - z^2 X) \wedge Y$.

So let us consider a Hamiltonian function, and the vector field

$$((X - z^2 Z) \wedge Y)\, \mathrm{d}H = ((X - z^2 Z) \wedge P\, \mathrm{d}[A]^1_2)\, \mathrm{d}H$$

which is given by

$$\{H, [A]^1_2\}_{\mathcal{P}} \cdot (X - z^2 Z) - \mathrm{Lie}_{X - z^2 Z}(H) \cdot \mathcal{P}\, \mathrm{d}[A]^1_2. \tag{2.16}$$

Since $\frac{\partial [A]^1_2}{\partial A} = \mathsf{K}$, a direct substitution into (2.6) shows that the vector field $Y = \mathcal{P}\, \mathrm{d}[A]^1_2$ is given by

$$\dot{A} = [A, \mathsf{K}] \qquad \dot{S} = [S, \mathsf{K}] \qquad \dot{B} = [B, \mathsf{K}]$$

that is, the second summand in equation (2.16) gives rise to Lax equations.

The vector field $X - z^2 Z$ is given by

$$\dot{A} = \mathsf{J} \qquad \dot{S} = 0 \qquad \dot{B} = -z^2 \mathsf{J}$$

so that, along its flows, one obtains

$$\dot{L}(z) = (z^2 A + zS + B)^{\bullet} = 0 \tag{2.17}$$

whence the validity of our assertion. Finally, notice that the deformation of the M-element of the Lax pair in (2.15) is explicitly given by

$$\hat{M}(z) - M(z) = \mathrm{Tr}\left(\mathsf{J}\cdot\left(\frac{\partial H}{\partial A} - z^2\frac{\partial H}{\partial B}\right)\right)\cdot\mathsf{K}. \tag{2.18}$$

□

We can finally make contact with the bi-Hamiltonian theory of the Kowalevski gyrostat discussed in section 5 of [8]. Indeed, the following proposition holds.

Proposition 2.5. *The Poisson pencil P_z defined on $\mathcal{M}^3_\Sigma$ restricts to the ten-dimensional submanifold $\mathcal{K} \subset \mathcal{M}^3_\Sigma$ defined by the equations*

$$[A]^1_2 = \gamma = \text{const} \qquad [B]^1_4 = [B]^1_3 = 0 \qquad [B]^1_2 = [B]^3_4 \tag{2.19}$$

and it coincides (after a suitable change of coordinates) with the Poisson pencil of equations (5.4a) and (5.4b) of [8]. The restriction of the Lax matrix $L(z)$ coincides as well with the Bobenko–Reyman–Semenov-Tian-Shansky one for the Kowalevski gyrostat.

Proof. We will divide the proof into two parts, since the nature of the 'constraints' (2.19) is different. First we consider $[A]^1_2 = \gamma$. It turns out that $[A]^1_2$ is a common Casimir of P and R. Indeed, this follows immediately noticing that

$$P\,\mathrm{d}[A]^1_2 = \mathcal{P}\,\mathrm{d}[A]^1_2 + Y([A]^1_2)\cdot X - X([A]^1_2)\cdot Y$$
$$R\,\mathrm{d}[A]^1_2 = \mathcal{R}\,\mathrm{d}[A]^1_2 + Y([A]^1_2)\cdot Z - Z([A]^1_2)\cdot Y$$

and that $X([A]^1_2) = 1$, $Z([A]^1_2) = 0$, $Y([A]^1_2) = \{[A]^1_2, [A]^1_2\}_P = 0$. So the Poisson pencil P_z restricts to any submanifold $[A]^1_2 = \gamma$. The other three constraints are more peculiar, since $[B]^1_4$, $[B]^1_3$, $[B]^1_2 - [B]^3_4$ are not common Casimirs. However, a careful analysis of the form of P_z on the submanifold $\mathcal{K}$ shows that all vector fields that are Hamiltonian w.r.t. either P or R are tangent to $\mathcal{K}$, so the assertion follows. □

Finally, we can make explicit contact with the formulation of the Kowalevski system already present in the literature. It is sufficient to parametrize $\mathcal{K}$ with the set of coordinates

$$\begin{array}{llll} l_1 = [A]^1_4 & l_2 = [A]^1_3 & l_3 = 1/2([A]^3_4 - \gamma) & x = -[B]^1_2 = -[B]^3_4 \\ g_1 = 1/2([S]^1_1 - [S]^3_3) & & g_2 = 1/2([s]^1_2 + [S]^3_4) & g_3 = [S]^1_3 \\ h_1 = 1/2([S]^1_2 - [S]^3_4) & & h_2 = -1/2([S]^3_3 + [S]^1_1) & h_3 = [S]^1_4 \end{array}$$

to recast the Lax matrix in its 'standard' form

$$L(\lambda) = z^3\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} + z^2\begin{bmatrix} 0 & -\gamma & -l_2 & -l_1 \\ \gamma & 0 & l_1 & -l_2 \\ l_2 & -l_1 & 0 & -2l_3-\gamma \\ l_1 & l_2 & 2l_3+\gamma & 0 \end{bmatrix}$$
$$+ z\begin{bmatrix} g_1-h_2 & g_2+h_1 & g_3 & h_3 \\ g_2+h_1 & -g_1+h_2 & h_3 & -g_3 \\ g_3 & h_3 & -g_1-h_2 & g_2-h_1 \\ h_3 & -g_3 & g_2-h_1 & g_1+h_2 \end{bmatrix} + \begin{bmatrix} 0 & x & 0 & 0 \\ -x & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & -x & 0 \end{bmatrix} \tag{2.20}$$

and obtain the Poisson tensors of [8], section 5.

3. The Kowalevski gyrostat as a GZ system

The 'ideology' of the GZ theory for integrable systems defined on a bi-Hamiltonian manifold $(\mathcal{M}, P_\lambda)$ essentially relies on the study of the Casimir functions of the Poisson pencil P_λ, to be found as a series (eventually, for the case of the so-called *flat* Poisson pencils, a *polynomial*) in the parameter λ of the pencil.

Standard procedures from the theory of Lenard–Magri sequences show that all the coefficients of these polynomials (or series) mutually commute. Let us now apply these ideas to the Kowalevski systems.

Since the Poisson pencil we have at our disposal is associated with a Lax scheme, it is natural to consider ad-invariant polynomials as sources of mutually commuting Hamiltonians. Indeed the following proposition holds.

Proposition 3.1. *Let us consider* $\det(L(z) - \mu I) = \mu^4 + \hat{C}_2\mu^2 + \hat{C}_0$. *Then* $\hat{C}_0$ *and* $\hat{C}_2$ *are polynomial Casimirs of* P_z.

Proof. The conclusion easily follows from the following two facts.

(1) The elementary ad-invariant polynomials of $\mathfrak{sp}(4)$, that is, $\phi_1 = \mathrm{Tr}(L^2)$ and $\phi_2 = \mathrm{Tr}(L^4)$, satisfy $P_z(\mathrm{d}\phi_i) = 0$, since the bi-Hamiltonian flows are represented by a Lax equation;

(2) $\det(L(z) - \mu I)$ depends on z^2 alone.

Furthermore, looking at the explicit form (2.20) of the Lax matrix, one can see that the coefficients $\hat{C}_0$, $\hat{C}_2$ of the characteristic polynomial of $L(z)$ are given by

$$\begin{aligned}\hat{C}_2 &= -4z^6 + H_0z^4 + H_1z^2 + H_2\\ \hat{C}_0 &= -4\gamma^2 z^{10} + \sum_{i=1}^{5} K_{i-1}z^{10-2\mathrm{i}}.\end{aligned} \tag{3.1}$$

So, discarding the leading constant terms, we obtain two polynomial Casimirs, $C_2 = \hat{C}_2 + 4z^6$, $C_0 = \hat{C}_0 + 4\gamma^2 z^{10}$. □

However, this is not the whole story. Since the manifold $\mathcal{K}$ is ten dimensional, and the rank of P_z is equal to *eight* (for generic z and at generic points of $\mathcal{K}$), not all the commuting Hamiltonians $H_1, \ldots, K_5$ can be functionally independent. In other words, the Lenard sequence obtained from C_2 comprises two vector fields X_1 and X_2, and that obtained from C_2 comprises four vector fields $\tilde{X}_i$, $i = 1, \ldots, 4$. However, the dimension of the span $\langle X_1, \ldots, \tilde{X}_4\rangle$ is equal to the expected number $4 = (\dim\mathcal{M} - \operatorname{corank} P_z)/2$. Indeed the following proposition holds.

Proposition 3.2. *The degree 4 polynomial*

$$\begin{aligned}C_1 &= z^4 J_0 + z^2 J_1 + J_2\\ &= z^4\left(K_0 - \frac{H_0{}^2}{4}\right) + z^2\left(K_1 - \frac{H_0H_1}{2}\right) + \left(K_2 - \frac{H_2H_0}{2} - \frac{H_1{}^2}{4}\right)\end{aligned} \tag{3.2}$$

is another Casimir of the Poisson pencil P_z*, functionally independent of* C_2*. So the two-field Kowalevski gyrostat has the structure of a GZ manifold of rank 2 and type* $(5, 5)$ *(see figure 1).*

$$\begin{tikzcd}
\mathbf{0} & & & & \mathbf{0} & & & \\
\mathrm{dH}_0 \arrow[u, "\mathbf{R}"] \arrow[r, "\mathbf{P}"] & \mathbf{X}_1 & & & \mathrm{dJ}_0 \arrow[u, "\mathbf{R}"] \arrow[r, "\mathbf{P}"] & \mathbf{Y}_1 & & \\
& \mathrm{dH}_1 \arrow[u] \arrow[r] & \mathbf{X}_2 & & & \mathrm{dJ}_1 \arrow[u] \arrow[r] & \mathbf{Y}_2 & \\
& & \mathrm{dH}_2 \arrow[u] \arrow[r] & \mathbf{0} & & & \mathrm{dJ}_2 \arrow[u] \arrow[r] & \mathbf{0}
\end{tikzcd}$$

Figure 1. The Lenard sequences for the two-field Kowalevski gyrostat.

$$\begin{tikzcd}
\mathbf{0} & & & \mathbf{0} & & \\
\mathrm{dH}_{0|\mathcal{S}} \arrow[u, "\mathbf{R}^D"] \arrow[r, "\mathbf{P}^D"] & \mathbf{X}_1 & & \mathrm{dJ}_{0|\mathcal{S}} \arrow[u, "\mathbf{R}^D"] \arrow[r, "\mathbf{P}^D"] & \mathbf{Y}_1 & \\
& \mathrm{dH}_{1|\mathcal{S}} \arrow[u] \arrow[r] & \mathbf{0} & & \mathrm{dJ}_{1|\mathcal{S}} \arrow[u] \arrow[r] & \mathbf{0}
\end{tikzcd}$$

Figure 2. The Lenard sequence for the Kowalevski top.

3.1. The reduction to the Kowalevski top

In this final section we will study the corresponding issues relative to the Kowalevski top, which is obtained from the system studied in the previous section, freezing the extra degrees of freedom ($\vec{h}$ and x) to zero[3]. We shall see that it is possible to induce a bi-Hamiltonian structure on the phase space $\mathcal{S}$ of the Kowalevski top, and that the Lenard sequences induce corresponding (shorter) Lenard sequences on $\mathcal{S}$. To do this, it is necessary to briefly illustrate a slightly unconventional point of view on the notion of Dirac brackets[4].

3.1.1. Dirac brackets. Let (M, P) be a Poisson manifold, and let us fix a distinguished set of functions ϕ_i, $i = 1, \ldots, k$, with $k < n = \dim(M)$. Suppose that

$$\Phi_0 := \{\phi_1(x) = \bar{\phi}_1, \ldots, \phi_k(x) = \bar{\phi}_k\}$$

defines a cosymplectic foliation on (M, P), that is, the matrix $\{\phi_i, \phi_j\} = G^{ij}$ is (generically) non-degenerate.

The bracket P^{D} (as a Poisson bracket on the manifold M) defined by the usual Dirac formula,

$$\{f, g\}_{\mathrm{D}} = \{f, g\} - \sum_{i,j} \{f, \phi_i\} G^{-1}_{ij} \{\phi_j, g\} \tag{3.3}$$

[3] For simplicity, the gyrostatic term γ will be set to zero as well.

[4] We learned this picture from conversations with B Dubrovin, that are herewith gratefully acknowledged.

has the following properties.

(1) The functions $\phi_i, i = 1, \ldots, k$ are Casimir functions for the bracket P^{D}.
(2) For every leaf S of the foliation Φ, the bracket P^{D} induces a corresponding Poisson bracket P_S^{D}; the immersion of (S, P_S^{D}) in (M, P^{D}) is a Poisson map. This means that, if $(\cdot)_S$ denotes restriction to S, we have the equality

$$P_S^{\mathrm{D}}\, \mathrm{d}_S f_S = P^{\mathrm{D}}\, \mathrm{d} f. \tag{3.4}$$

Notice that, thanks to this property, we will be allowed to avoid writing the subscript $(\cdot)_S$ in most of the computations performed in the next section.

3.1.2. Dirac Poisson brackets for the Kowalevski top. The phase space of the Kowalevski top is the submanifold $\mathcal{S} := \{x = 0, \vec{h} = 0, \gamma = 0\}$, inside the phase space $\mathcal{K}$ of the Kowalevski gyrostat in two fields. We have seen that $\mathcal{K}$ is endowed with the Poisson pencil $P_\lambda = P - \lambda R$ ($\lambda = z^2$); the GZ geometry of $(\mathcal{K}, P_\lambda)$ is encoded in the two Casimir functions

$$C_2 = H_0\lambda^2 + H_1\lambda + H_2 \qquad C_1 = J_0\lambda^2 + J_1\lambda + J_2$$

that give rise to the four bi-Hamiltonian vector fields X_1, X_2, Y_1, Y_2.

The following properties can be ascertained by means of direct calculations.

(1) The Dirac procedure can be applied for P_λ, with the foliation associated with the functions $\{\phi_1, \ldots, \phi_4\} = \{x, \vec{h}\}$. Actually, it turns out that the matrix G_λ of Poisson brackets of the constraints is the following:

$$\lambda \begin{bmatrix} 0 & h_2 - g_1 & -h_1 - g_2 & -g_3 \\ -h_2 + g_1 & 0 & -l_3 & l_2 \\ h_1 + g_2 & l_3 & 0 & -l_1 \\ g_3 & -l_2 & l_1 & 0 \end{bmatrix}.$$

This means that, outside the subvariety[5] Δ defined by

$$\det(G) = (\vec{g} \cdot \vec{l} + (\vec{h} \wedge \vec{l})_3)^2 = 0$$

we have that the Dirac procedure induces a new Poisson pencil on $\mathcal{K}$ (and correspondingly on $\mathcal{S}$), which we call $P_\lambda^{\mathrm{D}} = P^{\mathrm{D}} - \lambda R^{\mathrm{D}}$.
(2) All the vector fields X_i, Y_i are tangent to $\mathcal{S}$, and X_2 and Y_2 vanish there.
(3) The Hamiltonians H_2 and J_2 vanish identically on $\mathcal{S}$. Actually, the Casimir polynomials restrict to

$$\begin{aligned} C_{2|\mathcal{S}} &= (-4\,g_1 + 4\,l_3{}^2 + 2\,l_2{}^2 + 2\,l_1{}^2)z^4 + (-2\,g_3{}^2 - 2\,g_2{}^2 - 2\,g_1{}^2)z^2 \\ C_{1|\mathcal{S}} &= (8\,g_1 l_3{}^2 + 4\,g_2{}^2 + 8\,l_1{}^2 g_1 + 8\,l_1 g_2 l_2 - 4\,l_3{}^2 l_2{}^2 - 4\,l_3{}^2 l_1{}^2 - 4\,l_3{}^4)z^4 \\ &\quad + (4\,g_2{}^2 l_2{}^2 + 4\,g_1{}^2 l_1{}^2 + 8\,g_3 l_3 g_2 l_2 + 8\,l_1 l_3 g_1 g_3 + 8\,g_1 l_1 g_2 l_2 + 4\,l_3{}^2 g_3{}^2)z^2. \end{aligned}$$

(4) On the Kowalevski top submanifold $\mathcal{S}$, (the restriction of) P^{D} coincides with (the restriction of) P.

Finally, we are in a position to prove the following.

Proposition 3.3. *The flows of the Kowalevski top fill in a suitable pair of Lenard sequences with respect to the Dirac brackets P^{D}, R^{D}.*

[5] From now on we will implicitly work outside Δ to avoid cumbersome notations.

Proof. Let us consider the two Lenard sequences associated with P_λ. If we restrict them to $\mathcal{S}$, thanks to property 4 above, we can safely assert that

$$P^{\mathrm{D}}(\mathrm{d}H_1) = 0 \qquad P^{\mathrm{D}}(\mathrm{d}H_0) = X_1$$
$$P^{\mathrm{D}}(\mathrm{d}J_1) = 0 \qquad P^{\mathrm{D}}(\mathrm{d}J_0) = Y_1.$$

We simply have to prove that $R^{\mathrm{D}}(\mathrm{d}H_1) = X_1$, $R^{\mathrm{D}}(\mathrm{d}J_1) = Y_1$ and that $R^{\mathrm{D}}(\mathrm{d}H_0) = R^{\mathrm{D}}(\mathrm{d}J_0) = 0$, filling the missing vertical arrows in the Lenard sequence. All the proofs are done in the same way, so we will do this for X_1 only. To this end it is useful to represent R^{D} as follows: let $Z_i = R\,\mathrm{d}\phi_i$ and denote by Γ the inverse matrix to G. Then the Dirac bracket relative to R^{D} is represented by the Poisson tensor

$$R^{\mathrm{D}} = R - \sum_{i,j} \Gamma_{i,j} Z_i \wedge Z_j. \tag{3.5}$$

We have to compute $R^{\mathrm{D}}(\mathrm{d}H_1)$; so from (3.5) we have

$$R^{\mathrm{D}}(\mathrm{d}H_1) = R(\mathrm{d}H_1) - \sum_{i,j} \Gamma_{ij}(\mathrm{Lie}_{Z_i}(H_1)\cdot Z_j - \mathrm{Lie}_{Z_j}(H_1)\cdot Z_i).$$

The first term in the right-hand side of this equation is exactly X_1; moreover, in the sum over i, j, each element vanishes: indeed, we have, since $Z_i = R\,\mathrm{d}\phi_i$,

$$\mathrm{Lie}_{Z_i}(H_1) = \langle\, \mathrm{d}H_1, R\,\mathrm{d}\phi_i\rangle = -\langle\, \mathrm{d}\phi_i, R\,\mathrm{d}H_1\rangle = -\langle\, \mathrm{d}\phi_i, X_1\rangle$$

and it vanishes since X_1 is tangent to $\mathcal{S}$. This concludes our proof. □

Acknowledgments

The author is indebted to F Magri and I D Marshall for collaboration at the early stages of this work, and to B Dubrovin for discussions. Also, discussions with C Morosi and G Tondo on the possible role of the Dirac reduction procedure for the Kowalevski top are herewith acknowledged. This work was partially supported by the GNFM of the Italian CNR and by the Italian MURST under the *cofin99* project 'Geometry of integrable systems'.

References

[1] Adler M and van Moerbeke P 1988 The Kowalevski and Hénon–Heiles motions as Manakov geodesic flows on $SO(4)$—a two dimensional family of Lax pairs *Commun. Math. Phys.* **113** 659–700

[2] Belokolos E D, Bobenko A I, Enol'skii V Z, Its A R and Matveev V B 1994 *Algebro-Geometric Approach to Nonlinear Integrable Equations (Springer Series in Nonlinear Dynamics)* (Berlin: Springer)

[3] Bobenko A I, Reyman A G and Semenov-Tian-Shansky M A 1989 The Kowalevski top 99 years later: A Lax pair, generalisations and explicit solutions *Commun. Math. Phys.* **122** 321–54

[4] Gel'fand I M and Zakharevich I 1993 On the local geometry of a biHamiltonian structure *The Gel'fand Mathematical Seminars 1990–92* ed L Corwin *et al* (Boston: Birkhäuser) pp 51–112

[5] Gel'fand I M and Zakharevich I 1999 Webs, Lenard schemes, and the local geometry of bi-Hamiltonian Toda and Lax structures *Selecta Math.* at press
(Gel'fand I M and Zakharevich I 1999 *Preprint* math-ag/9903080)

[6] Haine L and Horozov E 1987 A Lax pair for Kowalevski's top *Physica* D **29** 173–80

[7] Magnano G and Magri F 1991 Poisson–Nijenhuis structures and Sato hierarchy *Rev. Math. Phys.* **3** 403–66

[8] Marshall I D 1998 The Kowalevski top: its r-matrix interpretation and bihamiltonian formulation *Commun. Math. Phys.* **191** 723–34

[9] Reyman A G and Semenov-Tian-Shansky M A 1987 Lax representation with a spectral parameter for the Kowalevski top and its generalizations *Lett. Math. Phys.* **14** 55–62

[10] Reyman A G and Semenov-Tian-Shansky M A 1994 Group-theoretical methods in the theory of finite-dimensional integrable systems *Encyclopedia of Mathematical Sciences (Dynamical Systems VII)* vol 16, ed V I Arnol'd and S P Novikov (Berlin: Springer)

[11] Ugaglia M 1998 On the biHamiltonian structure of the Kowalevski top *SISSA preprint* 64/98/FM